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On the Isolation of the Roots of a Non-Linear Operator*

A. A. GOLDSTEIN

Department of Mathematics, University of Washington, Seattle, Washington 98105

F. C. JOHNSON

Boeing Scientific Research Laboratories, Seattle

AND

JURGEN SPIESS

University of Hamburg, Hamburg

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Certain problems in optimal control may be formulated by the theory of Lagrange multipliers. Stationary controls are thus roots of an operator equation. See, e.g., Ref. [1, 2]. We give below a measure of the isolation of these roots.

We will follow the notational convention that $f'(x)$ represents the Gateaux derivative of f evaluated at x, and that $f'(x, h)$ is the differential of f at x in the direction h.

THEOREM. *Let G be a Gateaux differentiable mapping from an open convex subset C of a normed linear space N to a Hilbert space E. Assume that G has a root z in C, that G' satisfies $\| G'(z, x - z)\| \geqslant \mu \| x - z \|$ and $\| G'(z, x - z) - G'(y, x - z)\| \leqslant L \| x - z \| \| y - z \|$ for all $x, y \in S$ and some positive μ and L. Here S denotes the set $\{x \in C: \| x - z \| < \mu/L$. Then z is the only root of G in S.*

As an aside, we observe that since G is actually Frechet differentiable at z, the root z can easily be seen to be isolated.

Proof. Let $\phi(x) = \frac{1}{2}\langle G(x), G(x)\rangle$. We have $\phi(z) = 0 = \phi'(z)$ and $\phi(x) = \int_0^1 \phi'(z + t(x - z), x - z)\, dt$.

We show that $\phi(x) > 0$ for all $x \in S - \{z\}$.

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Let $h = x - z$ and $y = z + th$ with $0 < t \leqslant 1$; then

$$\begin{aligned}\phi'(y, h) &= \langle G(y), G'(y, h)\rangle \\ &= \langle G'(y, h), G'(y, h)\rangle + \langle G(y) - G(z) - G'(z, h), G'(y, h)\rangle \\ &\quad + \langle G'(z, h) - G'(y, h), G'(y, h)\rangle \\ &\geqslant \{\| G'(y, h)\| - \| G(y) - G(z) - G'(z, h)\| \\ &\quad - \| G'(z, h) - G'(y, h)\|\} \, \| G'(y, h)\|. \end{aligned} \tag{3}$$

We now estimate:

(a) Choose $f \in E^*$, such that $\| f \| = 1$ and

$$\begin{aligned}&\| G(y) - G(z) - G'(z, h)\| \\ &\quad = f(G(y) - G(z) - G'(z, h)) = f(G(y)) - f(G(z)) - f(G'(z, h)) \\ &\quad = f(G'(\xi, h)) - f(G'(z, h)) = f(G'(\xi, h) - G'(z, h)) \\ &\quad \leqslant \| G'(\xi, h) - G'(z, h)\| \leqslant L \| \xi - z \| \, \| h \| \leqslant Lt \| h \|^2.\end{aligned}$$

In the above, $\xi = z + \tau h$ and $0 < \tau < t$.

(b) $\| G'(z, h) - G'(y, h)\| \leqslant L \| z - y \| \, \| h \| - Lt \| h \|^2$.

(c) Since $\| G'(z, h)\| - \| G'(y, h)\| \leqslant \| G'(z, h) - G'(y, h)\| \leqslant Lt \| h \|^2$, we have that

$$\| G'(y, h) \geqslant \| G'(z, h)\| - tL \| h \|^2 \geqslant (\mu - tL \| h \|) \| h \|.$$

Combining a, b, c, in (3), we get

$$\phi'(y, h) \geqslant \| h \|^2 (\mu - t\alpha)(\mu - 3t\alpha) \qquad (\alpha = L \cdot \| h \|);$$

hence

$$\begin{aligned}\phi(x) &= \int_0^1 \phi'(y, h)\, dt \geqslant \| h \|^2 \int_0^1 (\mu - t\alpha)(\mu - 3t\alpha)\, dt = \| h \|^2 \cdot (\mu - \alpha)^2 \\ &= \| x - z \|^2 \cdot (\mu - L \| x - z \|)^2.\end{aligned}$$

Thus $\phi(x) > 0$, if $x \in S - \{z\}$.

LEMMA. *Let H_1 and H_2 be Hilbert spaces and A be a bounded linear operator from H_1 onto H_2. Then AA^* has a bounded inverse.*

Proof. We first observe that the null space of A, $N(A)$, equals $\{0\}$. Take $l \in H_2$, $l \neq 0$, and set $y = A^*l$. Suppose that $y \in N(A)$ and $y \neq 0$. Since $l \neq 0$, there exists $x \in N^\perp(A)$ such that $Ax = l$. Thus we have

$$0 = \langle x, y\rangle = \langle x, A^*l\rangle = \langle Ax, l\rangle = \langle l, l\rangle,$$

a contradiction. Moreover, we have $N(A^*) = \{0\}$ and therefore $N(AA^*) = \{0\}$. Thus, AA^* is a one-to-one map. Since the range of A is closed, the range of A^* is closed [3, p. 488]. Therefore, the range of AA^* is closed. Suppose that the range R of AA^* is not H_2, and take $l \in R^{\perp}$, $l \neq 0$. Then $0 = \langle l, AA^*l\rangle = \langle A^*l, A^*l\rangle$, a contradiction. Therefore AA^* is onto, and by the open mapping theorem $(AA^*)^{-1}$ is bounded.

Remark. If $B\colon H_1 \to H_1$ and B has a bounded inverse, then ABA^* has a bounded inverse.

Application to Constrained Minimization

Let A be an operator from H_1 to H_2 and f a functional defined on H_1. We consider the problem of minimizing f on the set $S = \{x \in H_1 : A(x) = 0\}$.

Let z be a stationary point of f on S, i.e., $z \in S$, the Fréchet derivatives A' and f' exist at z and there exists $\Lambda \in H_2$ such that $f'(z, h) = \langle \Lambda, A'(z, h)\rangle$ for all $h \in H_1$. Let C_0 be a convex neighborhood of z and assume that there exist positive numbers $\alpha, \beta, \gamma, \psi, \Omega, \theta$ and ν such that the following conditions hold on C_0:

(i) f has a second Gateaux derivative which satisfies

$$\|f''(x) - f''(z)\| \leqslant \alpha \| x - z \|.$$

(ii) A has a second Gateaux derivative which satisfies

$$\| A''(x) - A''(z)\| \leqslant \beta \| x - z \|.$$

(iii) $\| A''(x)\| \leqslant \gamma$

(iv) Set $D(x) = A'(x)\, A'^*(x)$. Then $D^{-1}(x)$ exists and satisfies

$$\| D^{-1}(x) - D^{-1}(z)\| \leqslant \psi \| x - z \|$$

(v) $\| D^{-1}(x)\| \leqslant \Omega$.

(vi) $\| A'(x)\| \leqslant \Theta$.

(vii) $\|f'(x) - f'(z)\| \leqslant \nu \| x - z \|$.

Let $H = H_1^* \oplus H_2$ and $S_0 = C_0 \oplus H_2$. Let G be a mapping from S_0 to H defined by

$$G(p) = (f'(x) - \lambda \circ A'(x),\ A(x)),$$

where $p = (x, \lambda)$ and $\lambda \circ A'(x) \in H_1^*$ is defined by $(\lambda \circ A'(x))h = \langle A'(x, h), \lambda\rangle$. Note that a zero of G is a stationary point of f on S and conversely. Let $p_0 = (z, \Lambda)$ and define the operator $C \in B(H_1, H_1)$ by

$$Ch = f''(z, h) - \Lambda \circ A''(z, h).$$

We shall show that $G'(p_0)^{-1}$ is bounded. Let $\eta = \| G'(p_0)^{-1} \|$ and define

$$L = \alpha + \beta \| \Lambda \| + 3\gamma,$$
$$M = \nu\Omega\Theta + (\psi \| A'(z)\| + \gamma\Omega) \| f(z)\|.$$

THEOREM. *Assume that $A'(z)$ maps H_1 onto H_2 and that C^{-1} exists. Let N be a sphere centered at z with radius $r = 1/(1+M)\eta L$. Then z is a unique stationary point of f with respect to S on the set $N \cap C_0$.*

Proof. Let $\Delta p = (h, l)$. Then

$$G'(p, \Delta p) = (f''(x, h) - \lambda \circ A''(x, h) - l \circ A'(x), A'(x, h)).$$

We not first that $G'(p_0)$ maps H onto H. For, let $(a, b) \in H$. Then we seek $\Delta p \in H$ such that $G'(p_0, \Delta p) = (a, b)$. Since the operator $\lambda \circ A'(z)$ also has the representation $A'^*(z)\lambda$, then, dropping the z argument for simplicity, we are led to the equations

$$Ch - A'^*l = a,$$
$$A'h = b,$$

which has the solution

$$l = (A'C^{-1}A'^*)^{-1}(b - A'C^{-1}a),$$
$$h = C^{-1}(a + A'^*l),$$

where the existence of $(A'C^{-1}A'^*)^{-1}$ is established by the remark following the lemma.

It follows that $G'(p_0)$ is also one-to-one and the open mapping theorem implies that $G'(p_0)^{-1}$ is bounded. Since we have defined $\eta = \| G'(p_0)^{-1} \|$, we have that

$$\| G'(p_0, \Delta p)\| \geqslant \frac{1}{\eta} \| \Delta p \|.$$

A straightforward computation shows that G' is Lipschitz continuous at p_0. We have

$$\| G'(p_0, \Delta p) - G'(p, \Delta p)\| \leqslant (\alpha + \beta \| \Lambda \| + 2\gamma) \| x - z \| \| \Delta p \| + \gamma \| \Lambda - \lambda \| \| \Delta p \|$$
$$- L \| p - p_0 \| \| \Delta p \|.$$

Therefore, G' is Lipschitz continuous at p_0 with constant L. It follows then from the theorem that if $\| p - p_0 \| \leqslant 1/\eta L$ and $p \neq p_0$, then $G(p) \neq 0$.

Let $y \in C_0$ such that y is a stationary point of f on S, and let κ be the corresponding multiplier. Then (y, κ) satisfies

$$f'(y) = A'^*(y)\kappa,$$

and it follows from Lemma 1 that

$$\kappa = D^{-1}(y)\, A'(y) f'(y).$$

We have then

$$\begin{aligned} \| \Lambda - \kappa \| &= \| D^{-1}(z)\, A'(z) f'(z) - D^{-1}(y)\, A'(y) f'(y) \| \\ &\leqslant (\psi \| A'(z) \| \| f(z) \| + \gamma \Omega \| f(z) \| + \nu \Omega \theta) \| y - z \| \\ &= M \| y - z \|. \end{aligned}$$

Thus, we have that

$$\| p_0 - (y, \kappa) \| \leqslant (1 + M) \| y - z \|.$$

Hence, if $y \in N \cap C_0$, then $\| p_0 - (y, \kappa) \| \leqslant 1/L\eta$, and since G has a unique root on $N \cap C_0$, we must have $(y, K) = p_0$. Q.E.D.

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